

Dear Prof. Zhang,

We are sincerely grateful to our two anonymous reviewers. We adopted all their comments and hope that the manuscript is now close to their expectations. All changes are marked red.

Respectfully,

Mikhail Verbitsky and Michel Crucifix

Response to Anonymous Referee #1

Dear Anonymous Referee #1,

We are grateful for your insightful review that will help us to improve the manuscript. Your *verbatim* comments are below (in bold), each followed by our response.

General comments: This manuscript has the potential to provide an important contribution to Ice Age modelling.

Response: We are encouraged by your recognition of such important potential.

General comments (cont'd): Building on their previous work (Verbitsky & Crucifix, 2020; 2022), the authors investigate the physical similarity of different models using dimensional analysis. A key finding is that although all these models are structurally dissimilar, they share a dependence of the modeled periodicity on the dimensionless "V" parameter or the ratio between positive and negative feedbacks in the system.

Response: Your observation is correct: Indeed, finding the key similarity parameter that allowed us to quantitatively compare phenomenological models with the model that we nominated to serve as the proxy of the parent dynamical system is very important. Another key finding, that you have not mentioned, is that not all phenomenological models have actually "passed" the similarity test and therefore did not demonstrate a link with known physical assumptions. This finding has in our view philosophical-level consequences: successful replication of empirical time series is not sufficient to claim physical similarity with Nature.

Having said all this, I do think that the authors will need to address the issues discussed below before the manuscript will be ready for publication.

Specific comments: I have two major concerns that the authors will need to address in a compelling manner:

1) The authors make it seem as if they are comparing models to Nature when they are really comparing different models to each other. And yes, even the Verbitsky et al. (2018) model is not the same thing as Nature. The authors should be transparent about this, which starts with the title. I suggest changing it from "Do phenomenological dynamical paleoclimate models have physical similarity with Nature? Seeming, but not all of them" to something like "Structural similarities and differences between paleoclimate models of glacial-interglacial dynamics". Then, the Abstract and Introduction could be framed around questions such as "To which extent are different paleoclimate

models of glacial-interglacial dynamics physically similar?" and "Are there any shared dimensionless quantities playing key roles in all these models? If so, then finding values for these quantities should be a central objective of future research into glacial-interglacial dynamics."

Response: Your concern is fair and it is well understood. In fact, we tried hard to be very transparent in that regard: (a) We introduced the notion of the parent dynamical system (lines 63-69), (b) then we formulated requirements to it and introduced the VCV18 model as a candidate, a proxy, for the parent system (lines 108-113, 132-143), (c) we even suggested that “the nomination of the VCV18 model to serve as a proxy of the parent dynamical system can, indeed, be questioned, and the developments of better proxies should be encouraged” (lines 427-429). We agree with you though that there is still room for improvement, therefore...

Action: Your questions "To which extent are different paleoclimate models of glacial-interglacial dynamics physically similar?" and "Are there any shared dimensionless quantities playing key roles in all these models? If so, then finding values for these quantities should be a central objective of future research into glacial-interglacial dynamics" are well formulated and we will definitely discuss them in the introduction and conclusions sections of the revised paper. We are a bit reluctant though to accept your new title suggestion because, in our opinion, it shifts the focus of the paper. We believe that it is not so much about how similar models are, but about how similar are models, that are merely a statistical description of the data (phenomenological models), and a model that was derived from the basic laws of physics. We will make sure though that the Nature in the title will become quote-unquote “Nature”.

Done: New lines 2, 23-28, 508-510

2) The authors use the Buckingham pi theorem to answer the identify the dimensionless parameters affecting the period of the system in each of the models. All well and good, but what do we really learn from this about glacial-interglacial dynamics? I think much more insight could be gained if the authors would identify the actual relationships between the parameters and the period of the model systems (i.e., the psi and phi functions). It should not be too difficult to find decent approximations of these relationships through simulations, given that the models under consideration are rather simple and computationally cheap to run. Such an exercise would also give much deeper insight about the physical similarity between the different models. For example, the period may depend on "V" in two models, but the scaling may be V² in one model and V³ in the other. Then, one can ask where these different scalings originate from and whether any of these scalings could be tested against observational data.

Response: Your observation is correct – we do not provide the most explicit form of the scaling laws, specifically, we do not convert

$$\Pi = \Phi(\Pi_1, \Pi_2, \dots, \Pi_i, \dots, \Pi_m) \tag{AC1}$$

into

$$\Pi = \Pi_1^{\alpha_1} \Pi_2^{\alpha_2} \dots \Pi_m^{\alpha_m} \tag{AC2}$$

Also, we do not convert

$$\pi = \Psi(\pi_1, \pi_2, \dots, \pi_i, \dots, \pi_q) \tag{AC3}$$

into

$$\pi = \pi_1^{\beta_1} \pi_1^{\beta_2} \dots \pi_1^{\beta_q} \quad (\text{AC4})$$

We limited our scope to the demonstration that in some models

$$\Psi \approx \Phi \quad (\text{AC5})$$

when

$$\pi_i \approx \Pi_i, \quad (\text{AC6})$$

i.e., π_i -physics in the model is as significant as the Π_i -physics of “Nature”. It allowed us to “delineate a model that is merely a statistical description of the data, from a model that can be claimed to have a link with known physical assumptions” (lines 20-22). To our knowledge, *this result is novel*.

Importantly, *the condition (AC6) can also be considered as a constraint on the parameterization structure of a model, and as such, it is indeed also a new insight*.

Though we agree with you that the finding of explicit scaling laws (AC2) and (AC4) is an important exercise, we believe that this should be a separate study because it may be a bit more challenging than it seems to be. For example, gradual increase of the V -number in the VCV18 model includes a bifurcation from the obliquity period to the obliquity-period doubling (Verbitsky and Crucifix, 2020). Moreover, the V -number in the VCV18 model is a conglomerate similarity parameter, composed by 5 similarity parameters and, though the bifurcation is imminent, the critical value of the V -number that starts a bifurcation depends on which specific similarity parameters produce the V -number change. For example, the bifurcation that is caused by increased V -number due to the intensification of the positive feedback and caused by changes of the governing parameter responsible for the positive feedback may start at the critical V -number value that is different from the critical V -number value caused by the weakening of the negative feedback (Verbitsky, 2022). Therefore, to get a complete picture, the number of experiments needed for spanning the full parameter space can be overwhelming.

However, to address your concern, a few steps in this direction can be done, indeed, without much of computer power. First, we have run a few additional experiments with the VCV18-1 model:

$$\frac{dH}{dt} = \hat{a} + \kappa H^4 - c\theta \quad (22)$$

$$\frac{d\theta}{dt} = \frac{H^4 - \theta}{H/\hat{a}} \quad (23)$$

Specifically, we numerically measured the dimensionless period of auto-oscillations (P -scaling law) $\frac{P}{\tau} = \Phi(V)$, changing parameter c , and thus gradually changing the balance between positive and negative feedbacks. The results of these additional experiments are presented in Figure AC1 together with the P -scaling law for LP22 model $\frac{P}{\tau_g} = \Psi(V_0, V)$ that can be estimated in a very straightforward manner:

$$P = V_0 \tau_g + \tau_d, \text{ or}$$

$$\frac{P}{\tau_g} = V_0 + \tau_d/\tau_g, \text{ or}$$

$$\frac{P}{\tau_g} = V_0 + V \tag{AC7}$$

And finally, as an illustration, we also made a few experiments with the VCV18 model, gradually changing the V -number and measuring the corresponding P -scaling law. For this specific illustration, the V -number was modified by changing the strength of the positive feedback through the coefficient γ in the equation (10). The results are also presented in Figure AC1.

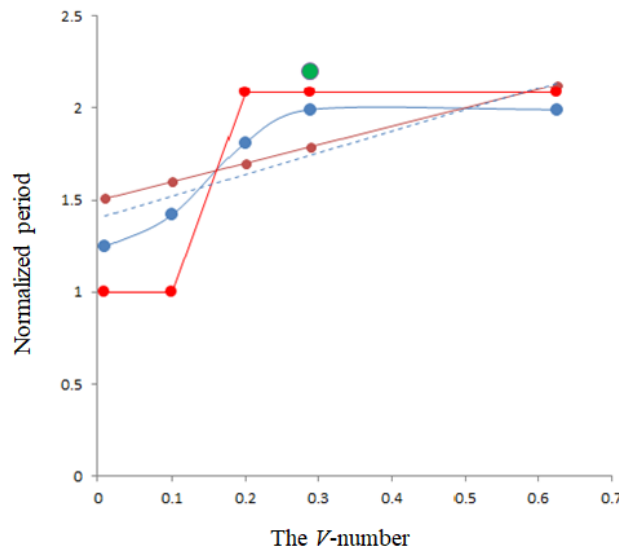


Fig. AC1. The P -scaling laws of VCV18-1 (blue, the dotted line is its trendline) and of LP22 (brown) models. The green dot marks the scaling law for the VDP model. The red line illustrates a bifurcation from the obliquity period to the obliquity-period doubling in the VCV18 model.

Interestingly, though the VCV18-1 scaling law $\frac{P}{\tau} = \Phi(V)$ and LP22 scaling law $\frac{P}{\tau_g} = \Psi(V_0, V)$ were produced absolutely independently, the LP22 scaling law coincides almost perfectly with the VCV18-1 scaling-law trendline. This closeness of the VCV18-1 and LP22 scaling laws supports our previous assertion that we can consider LP22 model as an approximation of the VCV18-1 (lines 407 – 415). The new figure also shows that better articulated positive feedbacks (i.e., increased V -number) lead to longer periods of both auto-oscillations and periods of the system response to the orbital forcings.

Action: We will include parts of the above conversation in the paper. In doing so, we will rely on our editor advice to reasonably constrain the divergence from the original scope.

Done: New Fig. 3 and new lines 428-434

Technical corrections

I. 16: "...similar with the..." -> "...similar to the..."

I. 55: "Dynamical system's theory tell us why..." -> "Dynamical Systems Theory tells us why..."

I. 59: "...phenomenon..." -> "...phenomenon..."

I 79: "...parameters, let say π_1 , is..." -> "...parameters, say π_1 , is..."

I. 375: "...according to pi-theorem:" -> "...according to the pi-theorem:"

Action: Thank you! All corrections will be taken care of.

Done: New lines 16, 62, 66, 86.

Response to Anonymous Referee #2

Dear Anonymous Referee #2,

We are grateful for your insightful review that will help us to improve the manuscript. Your *verbatim* comments are below (in bold), each followed by our response.

General comments: The authors aimed to measure the physical similarity between several phenomenological dynamical paleoclimate models and the real world. They proposed that a model should have a link with known physical assumptions, rather than a statistical description of proxy data. Overall, the manuscript is interesting and may have important implications for the future development/evaluation of phenomenological models.

Response: We are grateful for your vision that “the manuscript is interesting and may have important implications for the future development/evaluation of phenomenological models.”

Specific comments: As I am not an expert on this issue, I fully agree with the comments from Reviewer#1. However, I still have some comments that may be considered to improve the quality of the manuscript.

1. Figure 2 shows a specific case of the models used here. I am wondering how you choose the values of each parameter (e.g., why obliquity-period doubling is used?). Please clarify.

Response: We *do not prescribe* the obliquity-period doubling. Instead we calculate VCV18 and G23 models responses (with the reference values of parameters published in VCV18 and G23) to the external sinusoidal forcing of the obliquity period and find that in both cases the period of response is twice the forcing period. Most importantly, it happens with similarity parameters, both the ratio of astronomical forcing amplitude to the terrestrial growth rate and the positive-to-negative feedback ratio, having reasonably close values in both models. Therefore we conclude that there is physical similarity between VCV18 and G23 models in terms of the above similarity parameters.

Action: We will further clarify this in the text.

Done: New lines 490-492

2. The physical similarity is measured in the $\epsilon/\hat{\alpha}$ -V space. I am just wondering what are the limitations of the proposed method? For example, G23 model clearly differs from the VDP model in the $\epsilon/\hat{\alpha}$ -V space. Are there possibilities that G23 model shares similarities with VDP model in some aspects?

Response: Fig. 3 is just an attempt to present most important results visually in one diagram. Since it is 2-dimensional, it may demonstrate similarity (or its absence) in terms of only 2 similarity parameters. It was sufficient for us, since we found similarity between phenomenological and physical models in terms of two similarity parameters, the ratio of astronomical forcing amplitude to the terrestrial growth rate and the positive-to-negative feedback ratio. This space wouldn't be sufficient if more similarity parameters needed to be visually presented. For example, equation (21) shows that G23 model depends on the similarity parameter that is the ratio of the orbital period T to the internal timescale $1/a$. This similarity parameter is absent in VCV18 model. This absence of similarity is not apparent from Fig. 3, but it is discussed in lines 240-254.

Action: We will further clarify this in the text.

Done: Updated Fig. 4 and new lines 493-500

3. The positive-vs-negative feedback loops are important to understand the periodicity of glacial-interglacial cycles. Could you add additional discussions on the differences in the loops between each model?

Response: We agree with your observation. Indeed, "The positive-vs-negative feedback loops are important to understand the periodicity of glacial-interglacial cycles". While, these loops is the central part of VCV18-1 and LP22 models comparison (lines 407-415), in other cases such discussions, we admit, may be too brief.

Action: We will add additional discussions of the feedback-loops diagrams.

Done: New lines 267-270, 478-489

1 Do phenomenological dynamical paleoclimate models have physical similarity 2 with “Nature”? Seemingly, not all of them do.

3 Mikhail Y. Verbitsky^{1,2} and Michel Crucifix²

4 ¹Gen5 Group, LLC, Newton, MA, USA

5 ²UCLouvain, Earth and Life Institute, Louvain-la-Neuve, Belgium

6
7 Correspondence: Mikhaïl Verbitsky (verbitskys@gmail.com)

8
9 **Abstract.** Phenomenological models may be impressive in reproducing empirical time series but this
10 is not sufficient to claim physical similarity with Nature until comparison of similarity parameters is
11 performed. We illustrated such a process of diagnostics of physical similarity by comparing the
12 phenomenological dynamical paleoclimate model of Ganopolski (2023), the van der Pol model (as used
13 by Crucifix, 2013), and the model of Leloup and Paillard (2022) with the physically explicit Verbitsky *et*
14 *al* (2018) model that played a role of a reference dynamical system. We concluded that phenomenological
15 models of Ganopolski (2023) and of Leloup and Paillard (2022) may be considered to be physically
16 similar to the proxy parent dynamical system in some range of parameters, or in other words they may be
17 derived from basic laws of physics under some reasonable physical assumptions. We have not been able
18 to arrive to the same conclusion regarding the van der Pol model. Though developments of better proxies
19 of the parent dynamical system should be encouraged, we nevertheless believe that the diagnostics of
20 physical similarity, as we describe it here, should become a standard procedure to delineate a model that
21 is merely a statistical description of the data, from a model that can be claimed to have a link with known
22 physical assumptions.

23 The similarity parameters we advance here as the key dimensionless quantities are the ratio of the
24 astronomical forcing amplitude to the terrestrial ice-sheet mass influx and the so-called *V*-number that is
25 the ratio of the amplitudes of time-dependent positive and negative feedbacks. We propose to use
26 available physical models to discover additional similarity parameters that may play central roles in ice-
27 age rhythmicity. Finding values for these similarity parameters should become a central objective of
28 future research into glacial-interglacial dynamics.

29 30 31 1. Introduction.

32 A mathematical model that is constructed to understand a physical phenomenon must be simple
33 enough, otherwise the interpretation of the modeling results may be as difficult as the interpretation of
34 direct observations. In that regard, even most sophisticated space-resolving models of global climate
35 provide, indeed, a simplified picture of the phenomenon, but a much more drastic degree of simplification
36 is required when we study climate on timescales of tens of thousands of years. Faced with this challenge,
37 Barry Saltzman used to be a proponent of the phenomenological approach “through the construction of
38 low-order models in which the full behavior is projected onto the dynamics of a reduced number of
39 ...highly aggregated variables...” (Saltzman, 2002). Phenomenological models of paleoclimate variability
40 have routinely been used to explain certain characteristics of glacial-interglacial cycles (e.g., Saltzman
41 and Maasch, 1991, Saltzman and Verbitsky, 1992, 1993, 1994, Paillard, 1998, Tziperman et al, 2006,
42 Crucifix, 2013, Kaufmann and Pretis, 2021, Talento and Ganopolski, 2021, Leloup and Paillard, 2022,
43 Ganopolski, 2023). The core principle of the phenomenological approach is to fit model-produced time
44 series to the observational time series. When this goal is achieved, it is tacitly assumed that there must be
45 some physical similarity between the phenomenological model and Nature. We believe though that the
46 assumption of physical similarity with Nature can be more rigorously challenged before the implications
47 of a phenomenological model are accepted.

48 In fluid dynamics, for example, the concept of physical similarity is the cornerstone of any judgement
49 built on model experimentations. Classical similarity parameters, which emerge from the analysis of

50 fundamental conservation laws, like the Reynolds number, the Peclet number, the Euler number, etc.,
 51 quantify the relative importance of different aspects of fluid flow. For an experimental or a numerical
 52 model to be relevant, it should have quantitatively the same similarity parameters as those of the natural
 53 phenomenon being considered. We will now apply this concept of physical similarity to dynamical
 54 paleoclimate systems.

55 As physicists, we might want to describe a phenomenon such as ice ages as “emerging from
 56 fundamental laws”. However, the fundamental laws that we know in physics dictate interactions between
 57 particles. Perhaps one of the greatest challenges of the physical approach to complex systems is to explain
 58 how Nature organizes billions of billions of particles in interaction to generate some predictable behavior
 59 even on very long time scales such as, precisely, glacial-interglacial cycles. The methods of statistical
 60 physics tell us how to define macroscopic variables to describe the collective behavior of particles
 61 submitted to a conservation constraint, and how the phenomenon of dissipation emerges as a consequence
 62 of statistical mixing in a chaotic system. Dynamical system’s theory tells us why we mainly see the most
 63 unstable modes of a system (Haken, 2006) and how time scale separation assumptions allow us to focus
 64 on a subset of the system’s variables. In a nutshell, the theories of mathematical and statistical physics
 65 make it legitimate to assume that there is a natural parent dynamical system with much fewer degrees of
 66 freedom than Avogadro’s number, and which has generated the phenomenon that we see.

67 What “much fewer” means is not a straightforward matter. It depends on what we describe as the
 68 phenomenon, and how fine-grained this description is. For example, the successions of glacial-interglacial
 69 cycles and the timing of deglaciations appear to follow fairly simple, predictable rules (Tzedakis et al.,
 70 2017). Hence, it is legitimate to assume that the physical parent dynamical system, which dictates the
 71 evolution of the macroscopic state of climate at the orbital time scale, can be reduced to a small number
 72 of degrees of freedom.

73 Specifically, we may suggest that this parent dynamical system is governed by n physical parameters
 74 a_i such that a dependent variable of interest, x , can be expressed as function

$$75 \quad x = \varphi(a_1, a_2, \dots, a_i, \dots, a_n) \quad (1)$$

77 If k parameters of $a_1, a_2, \dots, a_i, \dots, a_n$ are parameters with independent dimensions, then, according to π -
 78 theorem (Buckingham, 1914), in the dimensionless form, the phenomenon (1) can be described by
 79 $m = n - k$ adimensional similarity parameters $\Pi_1, \Pi_2, \dots, \Pi_i, \dots, \Pi_m$:

$$80 \quad \Pi = \Phi(\Pi_1, \Pi_2, \dots, \Pi_i, \dots, \Pi_m) \quad (2)$$

81 Two physical phenomena have physical similarity if both of them are described in the adimensional
 82 form by the same function $\Phi(\Pi_1, \Pi_2, \dots, \Pi_i, \dots, \Pi_m)$ and have identical numerical values of similarity
 83 parameters $\Pi_1, \Pi_2, \dots, \Pi_i, \dots, \Pi_m$, though numerical values of the governing parameters
 84 $a_1, a_2, \dots, a_i, \dots, a_n$ may be different (e.g., Barenblatt, 2003).

85 As we have already mentioned, our knowledge about a parent dynamical system is suggested to us by
 86 the presence of empirical time series. It means that one of the similarity parameters, say Π_1 , is
 87 adimensional time $\frac{t}{\tau}$ (t and τ are dimensional time and a timescale, correspondingly), and all other
 88 parameters $\Pi_2, \dots, \Pi_i, \dots, \Pi_m$ are fixed to specific values. Hence, an experimental time series (neglecting
 89 measure errors) can be described as

$$90 \quad \Pi = \Phi\left(\frac{t}{\tau}, \Pi_2, \dots, \Pi_i, \dots, \Pi_m\right) \quad (3)$$

92 If we created a model dynamical system such that it is governed by p governing parameters b_i

$$93 \quad x = \psi(b_1, b_2, \dots, b_i, \dots, b_p) \quad (4)$$

94 and r parameters of $b_1, b_2, \dots, b_i, \dots, b_p$ are parameters with independent dimensions, then, again,
 95 according to π -theorem, in the dimensionless form, the model can be described by $q = p - r$
 96 adimensional similarity parameters $\pi_1, \pi_2, \dots, \pi_i, \dots, \pi_q$:

$$97 \quad \pi = \Psi(\pi_1, \pi_2, \dots, \pi_i, \dots, \pi_q) \quad (5)$$

98 For a specific time series, and for a fixed set of parameters $\pi_2, \dots, \pi_i, \dots, \pi_q$, the model (5) can be
 99 presented as

$$100 \quad \pi = \Psi\left(\frac{t}{\tau}, \pi_2, \dots, \pi_i, \dots, \pi_q\right) \quad (6)$$

101 The essence of the phenomenological approach is to fit the function $\Psi\left(\frac{t}{\tau}, \pi_2, \dots, \pi_i, \dots, \pi_q\right)$ to the
 102 function $\Phi\left(\frac{t}{\tau}, \Pi_2, \dots, \Pi_i, \dots, \Pi_m\right)$ under the “best” set of parameters $\pi_2, \dots, \pi_i, \dots, \pi_q$, i.e. to equate the
 103 model time series $\Psi\left(\frac{t}{\tau}, \pi_2, \dots, \pi_i, \dots, \pi_q\right)$ and the natural, empirical, time series $\Phi\left(\frac{t}{\tau}, \Pi_2, \dots, \Pi_i, \dots, \Pi_m\right)$:

$$104 \quad \Psi\left(\frac{t}{\tau}, \pi_2, \dots, \pi_i, \dots, \pi_q\right) = \Phi\left(\frac{t}{\tau}, \Pi_2, \dots, \Pi_i, \dots, \Pi_m\right) \quad (7)$$

105 It is obvious that even if the goal (7) is achieved at every $\frac{t}{\tau}$ -point, we still cannot claim the model (6)
 106 to be physically similar to “Nature” (3) until we prove that $\pi_i = \Pi_i$, i.e., π_i -physics in the model is as
 107 significant as the Π_i -physics of Nature. Simply speaking, merely *matching a proposed phenomenological*
 108 *model with empirical data does not make a case for physical similarity* because it does not provide an
 109 evidence that it happens for the right reason, the reason being the similarity parameters of the right value,
 110 i.e., $\pi_i = \Pi_i$.

111 But how can we compare π_i -physics of the phenomenological model and Π_i -physics of Nature if the
 112 phenomenological models are not derived from the laws of physics? Though, indeed, phenomenological
 113 models have not been derived from the laws of physics, they are not completely ignorant of the physical
 114 content: they still have a physical, measurable variable, time; they also have orbital and terrestrial
 115 forcings as well as positive and negative feedbacks. If the parent dynamical system was formulated in
 116 terms of similarity parameters formed by the ratios of timescales and by the ratios of the forcings’ and
 117 feedbacks’ amplitudes, then the comparison with phenomenological models that also use time scales and
 118 forcing and feedback ratios would be possible. The VCV18 model (Verbitsky et al, 2018), is one such
 119 candidate (a proxy) for a parent dynamical system. VCV18 was *derived* from the scaled mass- and heat-
 120 balance equations of the non-Newtonian ice flow. Next, we will derive scaling laws and similarity
 121 parameters for three phenomenological models: (a) the model of Ganopolski (2023); (b) van der Pol
 122 model as it has been described by Crucifix (2013); and (c) the model of Leloup and Paillard (2022);
 123 G23, VDP and LP22, thereafter, respectively. Each of these models produces a specific
 124 function $\Psi(\pi_1, \pi_2, \dots, \pi_i, \dots, \pi_q)$. We then compare functions $\Psi(\pi_1, \pi_2, \dots, \pi_i, \dots, \pi_q)$ of these models
 125 with the corresponding function $\Phi(\Pi_1, \Pi_2, \dots, \Pi_i, \dots, \Pi_m)$ provided by VCV18 to recognize or reject the
 126 hypothesis of physical similarity with a proxy for the parent dynamical system.

127 Certainly, we cannot expect that the time series produced by G23, VDP, and LP22 models and by the
 128 VCV18 model are identical, and therefore these models will not be physically similar in the most rigorous
 129 sense of the equation (7). We will demonstrate though that the answer to the physical-similarity question
 130 is insightful if our dependent variable of interest x is not necessarily a time series but a time-independent
 131 attribute such as the period of glacial rhythmicity. All models of this study reproduce equally well ~ 100 -
 132 kyr period of the late Pleistocene glaciations. We will now evaluate if the similarity parameters involved
 133 in the corresponding equations (7) are quantitatively the same.

134
 135

136 **2. Method**

137

138 **2.1 VCV18 model as a proxy for a parent dynamical system.**

139

140 Deriving a low-order dynamical paleoclimate model that may be considered as a candidate (a proxy)
 141 parent dynamical system is not a trivial exercise. The “low-order” challenge means that out of the
 142 multitude of physical phenomena involved only few should be recognized as dominant ones, and the
 143 “dynamical” challenge means that the space-resolving properties should be sensibly reduced to some
 144 integrated variables. Accordingly, in developing VCV18 proxy parent dynamical system, we first
 145 assumed that ice ages can be explained by only two components of the global climate system, continental
 146 ice sheets and the ocean representing the rest of the climate. For an ice sheet we adopted mass,
 147 momentum, and heat conservation equations of a “thin” layer of homogeneous non-Newtonian ice, and
 148 the rest of the climate was represented by the energy-balance equation. To migrate from the three-
 149 dimensional to dynamical equations we used scaling analysis that provides simple mathematical
 150 statements that are consistent with the original physics. Accordingly, the VCV18 dynamical model of the
 151 ice-climate system is defined by the following set of equations:

152
$$\frac{dS}{dt} = \frac{4}{5} \zeta^{-1} S^{3/4} (\hat{a} - \varepsilon F - \kappa \omega - c\theta) \tag{8}$$

153
$$\frac{d\theta}{dt} = \zeta^{-1} S^{-1/4} (\hat{a} - \varepsilon F - \kappa \omega) \{ \alpha \omega + \beta [S - S_0] - \theta \} \tag{9}$$

154
$$\frac{d\omega}{dt} = -\gamma [S - S_0] - \frac{\omega}{\tau} \tag{10}$$

155

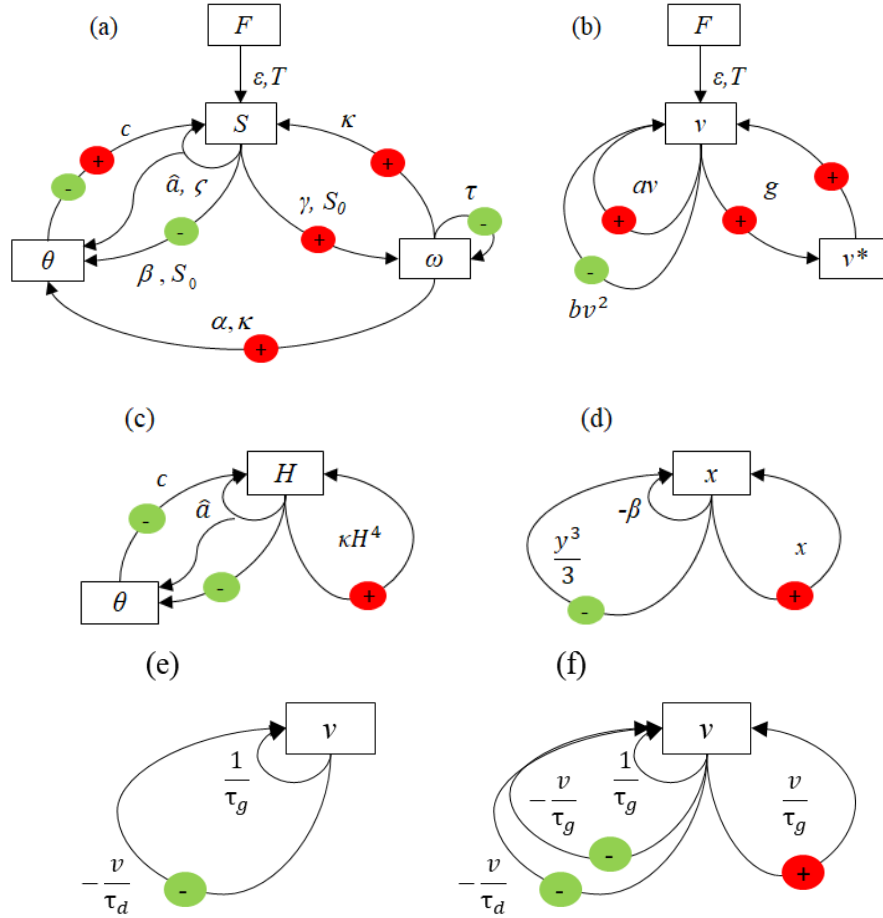
156 Here, S (m^2) is the area of glaciation, θ ($^{\circ}\text{C}$) is the basal ice-sheet temperature, and ω ($^{\circ}\text{C}$) is the global
 157 temperature of the rest of the climate. Equation (8) represents global ice balance $\frac{d(HS)}{dt} = AS$, where the
 158 ice thickness H is determined from the thin-layer approximation of ice flow, $H = \zeta S^{1/4}$, ζ is dimensional
 159 profile factor (Verbitsky and Chalikov, 1986) and $A = \hat{a} - \varepsilon F - \kappa \omega - c\theta$ is the surface mass influx.
 160 Equation (9) describes vertical ice temperature advection with a time scale $H/(\hat{a} - \varepsilon F - \kappa \omega)$, and
 161 equation (10) is the global energy-balance equation. The parameter \hat{a} (m s^{-1}) is the snow precipitation
 162 rate; F is normalized external forcing, i.e., mid-July insolation at 65°N (Berger and Loutre, 1991) of the
 163 amplitude ε (m s^{-1}); $\kappa \omega$ represents fast positive feedback from the global climate on ice-sheet mass
 164 balance; $c\theta$ is the ice discharge due to ice-sheet basal sliding incorporating (both delayed due to the
 165 vertical temperature advection) positive feedback from the global temperature, $\alpha \omega$, and a negative
 166 feedback of basal temperature reaction to the changes of ice geometry $\beta [S - S_0]$. Further, $-\gamma [S - S_0]$ is
 167 external forcing for global temperature (e.g., albedo); κ ($\text{m s}^{-1} \text{ } ^{\circ}\text{C}^{-1}$), c ($\text{m s}^{-1} \text{ } ^{\circ}\text{C}^{-1}$), α (adimensional), β
 168 ($^{\circ}\text{C m}^{-2}$) and γ ($^{\circ}\text{C m}^{-2} \text{ s}^{-1}$) are sensitivity coefficients; S_0 (m^2) is a reference glaciation area; and τ (s) is the
 169 global-temperature timescale.

170 Schematically, the dynamical system (8) – (9) is shown in Fig. 1(a). It can be seen that the dynamics
 171 of the VCV18 system is defined by the amplitude and periodicity of the orbital forcing, ε, T , by the
 172 terrestrial forcing \hat{a} , and by three feedback loops: the fast positive feedback, $-\kappa \omega$, and by two delayed,
 173 positive and negative feedbacks, combined in the term $-c\theta$. The dimensional analysis of the VCV18
 174 model has been performed previously (Verbitsky and Crucifix, 2020, 2021, Verbitsky, 2022a). It was
 175 revealed that its large-scale periodicity is generally governed by two dimensionless parameters: the ratio
 176 of the astronomical forcing amplitude ε to the terrestrial ice-sheet mass influx, $\Pi_2 = \varepsilon/\hat{a}$ and the so-called
 177 V -number, $\Pi_3 = V$ that is the ratio of amplitudes of time-dependent positive and negative feedbacks.
 178 Specifically, the period P of the VCV18 system response to the astronomical forcing of period T is of the
 179 form (hereafter called the “ P -scaling law”):

180 $\frac{P}{T} = \Phi\left(\frac{\varepsilon}{\hat{a}}, V\right)$ (11)

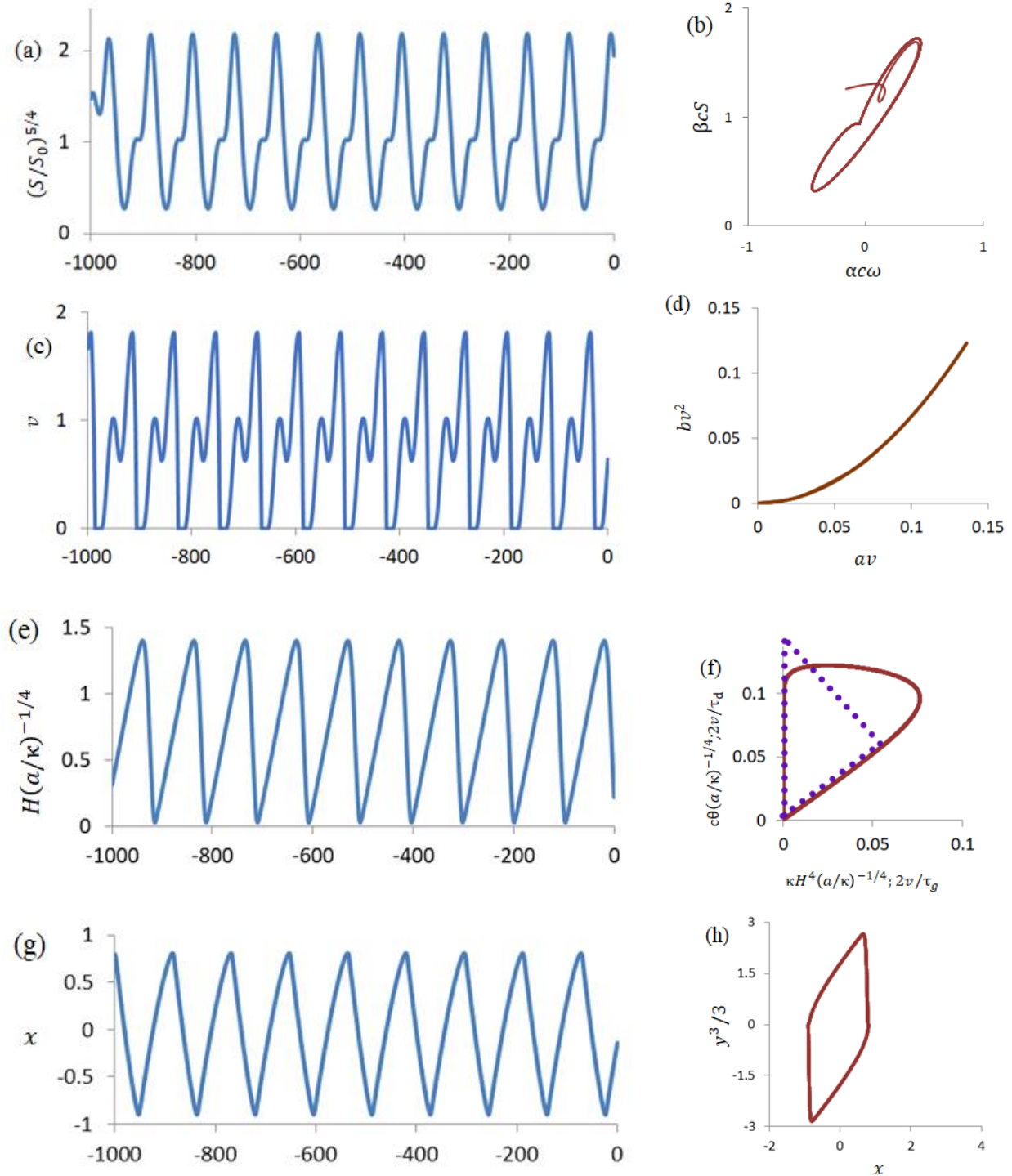
181 For $T = 40$ kyr, $\frac{\varepsilon}{\hat{a}} = 1.4$, $V = 0.7$, $\Phi = 2$ (obliquity-period doubling). The corresponding time series and
 182 positive-versus-negative feedback evolution are shown in Fig. 2 (a, b).

183



184

185 **Figure 1.** (a) The parent dynamical system VCV18 (Eqs. 8–10). Red circles mark positive feedback loops
 186 and green circles mark negative feedback loops; (b) The same for the G23 phenomenological model (Eqs.
 187 12-13); (c) The same for simplified system VCV18-1 (Eqs. 22–23); (d) The same for VDP model (Eqs.
 188 30-31); (e) The same for LP22 model (Eqs. 37-38), $I = 0$; (f) The same for LP22 model (Eqs. 44-45);



190

191 **Figure 2.** Time series (kyr BP) and corresponding positive-vs-negative feedback loops: (a, b) VCV18192 (Eqs. 8–10), $(\frac{S}{S_0})^{5/4}$ is normalized ice volume; (c, d) G23 (Eqs. 12-13); (e, f) VCV18-1 (Eqs. 22–23); all193 variables are normalized by characteristic ice thickness, $H' = (\hat{a}/\kappa)^{1/4}$; the dotted triangle corresponds to

194 LP22 (Eqs. 37-38) without astronomical forcing; (g, h) VDP (Eqs. 30-31).

195

196 2.2 G23 model.

197 The G23 model (specifically, Model 2, as it is annotated in Ganopolski, 2023) describes evolution of
 198 global ice volume v (adimensional) as a response to orbital forcing εF (F is normalized external forcing
 199 of the amplitude ε):

$$200 \frac{dv}{dt} = \frac{av - bv^2 - \varepsilon F + d}{1 - \delta g v^*} \quad (12)$$

$$201 \quad 202 v^* = \frac{1}{T^*} \int_{t-T^*}^t v(t') dt' \quad (13)$$

204 The term $\delta g v^*$ represents an additional positive feedback activated ($\delta = \delta_1 = 1$) when $\frac{dv}{dt} < 0$.

205 When $\frac{dv}{dt} \geq 0$, $\delta = \delta_2 = 0$. Graphically, the dynamical system (12) – (13) is shown in Fig. 1(b). We
 206 observe that the dynamics of the G23 system is, like VCV18, defined by the amplitude and periodicity of
 207 the orbital forcing, ε, T and by three feedback loops: two positive feedbacks, av, gv^* and by one negative
 208 feedback, $-bv^2$. Unlike VCV18, though, all feedbacks are instantaneous. We now review how these
 209 differences may be reflected in the corresponding P -scaling law.

210 For the purpose of dimensional analysis, we consider Eqs. (12) – (13) in dimensional form assuming
 211 the following dimensions for variables and parameters involved: t (s), v (m^3), a (s^{-1}), b ($\text{m}^{-3}\text{s}^{-1}$), ε (m^3s^{-1}), F
 212 is an adimensional function of the period T (s), d (m^3s^{-1}), δ_1, δ_2 (adimensional), g (m^{-3}), v^* (m^3). The
 213 period of the system response to the astronomical forcing is then a function of the following governing
 214 parameters:

$$215 \quad 216 P = \psi(a, b, \varepsilon, T, d, \delta_1, \delta_2, g) \quad (14)$$

218 For more explicit physical interpretation, instead of the parameter b , we will use parameter $\hat{a} = \frac{a^2}{b}$,
 219 (m^3s^{-1}), which is the mean growth rate. Also, for the reference values of parameters, provided by
 220 G23, $d \ll \varepsilon$, and, lastly, δ_1, δ_2 are constant. Therefore we can re-write (14) as:

$$221 \quad 222 P = \psi(a, \hat{a}, \varepsilon, T, g) \quad (15)$$

224 If we choose \hat{a}, T as parameters with independent dimensions, then according to π -theorem:

$$226 \frac{P}{T} = \Psi\left(\frac{\varepsilon}{\hat{a}}, T a, T g \hat{a}\right) \quad (16)$$

228 Let us now determine the V -number for G23 as the ratio of amplitudes of time-dependent positive and
 229 negative feedbacks. Obviously, such ratio should be completely defined by the internal (terrestrial) G23
 230 properties and therefore:

$$232 V = \lambda(a, b, g) \quad (17)$$

234 If we choose a and b as parameters with independent dimensions, then according to π -theorem:

$$236 V = \Lambda\left(\frac{ga}{b}\right) \quad (18)$$

238 We can also express g as a function of V :

$$240 \frac{ga}{b} = \Lambda^{-1}(V) \quad (19)$$

241
242 Accordingly, the P -scaling law of G23 can be written as:

$$243 \frac{P}{T} = \Psi \left[\frac{\varepsilon}{\hat{a}}, T\alpha, T\alpha\Lambda^{-1}(V) \right] \quad (20)$$

245
246 or

$$247 \frac{P}{T} = \Psi \left(\frac{\varepsilon}{\hat{a}}, \frac{T}{\tau}, V \right); \tau = 1/\alpha \quad (21)$$

249 We see that the scaling law (21) is different from the scaling law (11) because the Ψ – function of (21)
250 depends on T unlike the Φ – function of (11). There are only two scenarios for orbital periods to escape
251 the Ψ – function (or the Φ – function) in a scaling law. First, they may be excluded from the governing
252 equations when the main period of system’s variability is attributed to the internal, terrestrial, physics.
253 This is the case for VDP and LP22 models that will be considered later, but, definitely, it is not applicable
254 neither to G23 nor VCV18. The second scenario occurs when a system incorporates multiple parameters
255 encoding different time scales. The interplay of these parameters may create a situation when T -
256 dependent similarity parameters form jointly a T -independent conglomerate similarity parameter, giving a
257 system the so-called property of incomplete similarity (Barenblatt, 2003). This property has been
258 discovered for VCV18 (Verbitsky, 2022a). As the result, the Φ – function of the scaling law (11) does
259 not depend on orbital period ($\Phi = 2$ in the range of $T = 35 - 60$ kyr). Contrarily, as we have already
260 noted, G23 positive and negative feedbacks are instantaneous, G23 single ice-growth timescale $\tau \sim 1/\alpha$
261 does not have a “counterpart” for an interplay, and therefore the Ψ – function of (21) is period- T
262 dependent.

263 Nevertheless, for $T = 40$ kyr, $\frac{\varepsilon}{\hat{a}} = 1.6$, $V = 1.1$, $\Psi = 2$ (obliquity-period doubling). Hence, *only for*
264 *the obliquity forcing*, VCV18 and G23 models appear to be physically similar in regards of two,
265 **quantitatively close** similarity parameters, $\frac{\varepsilon}{\hat{a}}$, the ratio of the astronomical forcing amplitude ε to the
266 growth rate \hat{a} , and in terms of the V -number. The corresponding time series and positive-versus-negative
267 feedback evolution are shown in Fig. 2 (c, d). **Geometrically, the similarity between VCV18 and G23**
268 **models in terms of the V -number emerges from Fig. 2 (b) and Fig. 2 (d) as similar slopes of the**
269 **corresponding feedback loops, though, indeed, the instantaneous nature of G23 feedbacks makes their**
270 **“loop” asymptotically thin.**

271

272 **2.3 Simplified VCV18 model (VCV18-1) as a proxy for a parent dynamical system**

273

274 The next phase of our study is devoted to two phenomenological models, VDP and LP22 that have
275 100-kyr auto-oscillations independently of orbital forcing. To make the diagnostics of physical similarity
276 possible, we have to further simplify VCV18 system with several, physically reasonable, assumptions:

- 277 (a) Since the global-temperature timescale in equation (10) is much faster than other timescales
278 (orbital, ice accumulation, and ice-temperature advection), we assume that global temperature is
279 an instantaneous function of the glaciation forcing,
- 280 (b) In equation (9), we assume $\alpha = 0$ (for example, effect of increased global temperature is offset
281 by increased snow precipitation rate, see experiment D in the Appendix of VCV18), which
282 cancels the direct effect of climate on basal temperature,
- 283 (c) We rewrite equations (8) and (9) in terms of ice thickness $H = \zeta S^{1/4}$, and finally
- 284 (d) We attribute all system variability to terrestrial causes ($\varepsilon = 0$).

285 The simplified dynamical system then takes the following form:

286 $\frac{dH}{dt} = \hat{a} + \kappa H^4 - c\theta$ (22)

287

288 $\frac{d\theta}{dt} = \frac{H^4 - \theta}{H/\hat{a}}$ (23)

289

290 The physical meaning of all variables and governing parameters are the same as in equations (8) – (9), but
 291 the numerical values and dimensions of some parameters and variables are, indeed, different. Specifically,
 292 $t(s)$, $H(m)$, $\theta(m^4)$, $\hat{a}(m\ s^{-1})$, $\kappa(m^{-3}s^{-1})$, $c(m^{-3}s^{-1})$. The casual graph of the dynamical system (22) – (23) is
 293 shown in Fig. 1 (c).

294 The period of system variability is a function of three governing parameters:

295

296 $P = \varphi(\hat{a}, \kappa, c)$ (24)

297

298 If we choose \hat{a}, κ as parameters with independent dimensions, then according to π -theorem:

299

300 $\frac{P}{\tau} = \Phi\left(\frac{\kappa}{c}\right)$ (25)

301

302 $\tau = (\hat{a}^3 \kappa)^{-1/4}$

303

304 Parameters \hat{a}, κ, c lack in VDP and LP22 models, and therefore we transition to the V -number that must
 305 be a function of the same \hat{a}, κ, c parameters:

306

307 $V = \lambda(\hat{a}, \kappa, c)$ (26)

308

309 Since the V -number is adimensional, and \hat{a}, κ are parameters with independent dimensions, then
 310 according to π -theorem:

311

312 $V = \Lambda(\kappa/c)$ (27)

313

314 It also means that

315

316 $\frac{\kappa}{c} = \Lambda^{-1}(V)$ (28)

317

318 and we can finally present the VCV18-1 P -scaling law as:

319

320 $\frac{P}{\tau} = \Phi(V)$ (29)

321

322 In other words, the P -scaling law of the VCV18-1 system is fully defined by the balance between positive
 323 and negative feedbacks. For $\tau = 50$ kyr, $V = 0.63$, $\Phi = 2$. The corresponding 100-kyr-period auto-
 324 oscillations of the system (22) – (23) and its positive-versus-negative feedback loop are shown in Fig. 2
 325 (e, f).

326 2.4 VDP model

327 We now consider the VDP model, which is a variant of the historical van der Pol model (1922) used
 328 by De Saedeleer et al. (2013) and Crucifix (2013) to study synchronization properties of ice ages.

329 $\frac{dx}{dt} = \frac{-\beta - y}{\tau}$ (30)

330
$$\frac{dy}{dt} = \frac{\alpha}{\tau} \left(y - \frac{y^3}{3} + x \right) \quad (31)$$

331 Here all variables and parameters, except time and timescale τ , are adimensional. Variable x is a proxy for
 332 the glaciation, and variable y represents the rest of the climate. Since $\frac{\tau}{\alpha} \ll \tau$, for longer, glaciation-like
 333 processes, we can re-write equations (30) – (31) as:

334
$$\frac{dx}{dt} = \frac{1}{\tau} \left(-\beta + x - \frac{y^3}{3} \right) \quad (32)$$

335
$$y - \frac{y^3}{3} + x = 0 \quad (33)$$

336 The equation (33) defines the “critical manifold” (Guckenheimer et al, 2003). The system (32) – (33)
 337 describes VDP “slow” dynamics between two glacial-interglacial bifurcation points. To get VDP time
 338 series, we solve the non-idealized system (30) – (31), and we use the system (32) – (33) to visualize
 339 system’s positive, x , and negative, $\frac{y^3}{3}$ feedbacks. Schematically, the dynamical system (30) – (31) is
 340 shown in Fig. 1 (d). The period of system (30) – (31) variability is the function of two governing
 341 parameters:

342
 343
$$P = \psi(\tau, \beta) \quad (34)$$

344 Only parameter τ is dimensional, and therefore according to π -theorem:

345
$$\frac{P}{\tau} = \Psi(\beta) \quad (35)$$

346 The amplitudes of VDP variables are defined by the critical manifold (33) that does not contain
 347 parameters τ, β and therefore both x - and y -amplitudes do not depend on model parameters. Consequently
 348 the amplitudes of positive and negative feedbacks do not depend on them either. In fact, the x -amplitude
 349 in VDP model is always ~ 0.8 and y -amplitude is always 2. Therefore the ratio of the amplitude of the
 350 positive feedback, x , to the amplitude of the negative feedback, $\frac{y^3}{3}$, is always $V = 0.3$. In summary:

351
 352
$$V = const \quad (36)$$

353
 354 The property (36) makes VDP model to be fundamentally different from all other models in this
 355 study. In all other models, the V -number is the function of model’s governing parameters and, under
 356 different scenarios, it changes when parameters change. In the VDP model, the V -number is pre-defined
 357 by the model’s structure. Consequently, the scaling law (35) does not contain any V -number and thus it
 358 cannot match the scaling law (29). Therefore, *there is no physical similarity* between the VCV18-1 and
 359 the VDP models.

360 The auto-oscillations of the system (30) – (31) and positive-versus-negative feedback evolution are
 361 shown in Fig. 2 (g, h). For $\tau = 50$ kyr and $\beta = 0.3$, $\Psi = 2.2$. Fig. 2 (e) and Fig. 2 (g) show well why the
 362 phenomenological approach may be misleading. For the same internal timescale of 50 kyr, VCV18-1 and
 363 VDP models both produce asymmetrical (slow growth and fast retreat) glaciation time series with the
 364 respective periods P close to 100 kyr, but this occurs because of very different physics: in the VDP model
 365 the positive feedbacks are much weaker (as we already know, $V = 0.3$, always) than in the VCV18-1
 366 model ($V = 0.63$). Most importantly, this discrepancy cannot be changed, because the VDP model is
 367 rigid in this regard.

368
 369
 370

371 **2.5 LP22 model**

372

373 The LP22 model is described by two differential equations, first, for the growing ice volume,

374
$$\frac{dv}{dt} = -\frac{I}{\tau_i} + \frac{1}{\tau_g} \quad (37)$$

375 and another one for the diminishing ice volume

376
$$\frac{dv}{dt} = -\frac{I}{\tau_i} - \frac{v}{\tau_d} \quad (38)$$

377 Here, v and I are normalized ice volume and astronomical forcing, correspondingly; τ_i , τ_g , and τ_d are
 378 dimensional timescales. Additionally, if $I < I_0$ the system switches from equation (38) to equation (37),
 379 and if $I + v > V_0$, the system switches from equation (37) to equation (38). Though the original LP22
 380 model does not consider its evolution without astronomical forcing, oscillations still occur when $I = 0$ and
 381 the equation-switching conditions are, correspondingly, $v \leq V_1$ (V_1 is the minimal, interglacial, volume)
 382 and $v \geq V_0$. Schematically, the dynamical system (37) – (38) with $I = 0$ is shown in Fig. 1 (e). The period
 383 of auto-oscillations is a function of four parameters:

384
 385
$$P = \psi(V_0, \tau_g, V_1, \tau_d) \quad (39)$$

386 The parameter $V_1 \ll V_0$ can be settled as a constant, therefore:

387
$$P = \psi(V_0, \tau_g, \tau_d) \quad (40)$$

388 If we select τ_g as an independent-dimension parameter, then according to π -theorem:

389
$$\frac{P}{\tau_g} = \Psi(V_0, \tau_d/\tau_g) \quad (41)$$

390 The equation (37) describes a linear ice volume growth implying zero net feedback. This doesn't indicate
 391 the absence of feedbacks. Indeed, if we are ready to accept that LP22 model is more than just a successful
 392 fit to empirical data, then for the growing ice sheet, the equation (37) should be consistent with the
 393 dimensional total mass balance

394
 395
$$\frac{dv}{dt} = \hat{a}S \quad (42)$$

396
 397 i.e., changes of ice volume are equal to mass influx \hat{a} accumulated over its area S . If we multiply and
 398 divide $\hat{a}S$ by ice thickness H , the total mass balance (42) becomes

399
 400
$$\frac{dv}{dt} = \frac{v}{\tau_g} \quad (43)$$

401
 402 where $\tau_g = H/\hat{a}$. The equation (43) tells us that the positive feedback $\frac{v}{\tau_g}$ must be present in the growing
 403 ice sheet: it relates to the area growth with volume. Its absence in equation (37), therefore suggests that \hat{a}
 404 has another component that completely compensates for $\frac{v}{\tau_g}$, and yet another component that is inversely
 405 proportional to S (e.g., continentality effect). Hence, the system (37) – (38) must be written as follows to
 406 have physical meaning:

407

408 $\frac{dv}{dt} = \frac{1}{\tau_g} + \frac{v}{\tau_g} - \frac{v}{\tau_d}$ (44)

409 $\frac{dv}{dt} = -\frac{v}{\tau_d}$ (45)

410 Schematically, the dynamical system (44) – (45) is shown in Fig. 1 (f). The amplitude of the positive
 411 feedback is $\frac{V_0}{\tau_g}$ and the amplitude of the negative feedback is $\max\left\{\frac{V_0}{\tau_g}; \frac{V_0}{\tau_d}\right\}$. Since $\tau_g > \tau_d$, the V -number
 412 therefore is equal to:

413
 414 $V = \frac{\tau_d}{\tau_g}$ (46)

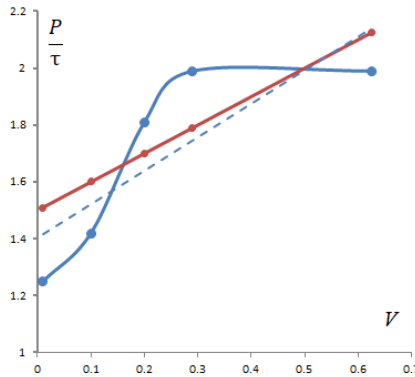
415 Accordingly, the P -scaling law (41) for LP22 model can be written as

416 $\frac{P}{\tau_g} = \Psi(V_0, V)$ (47)

417 For example, for $\tau_g = 50$ kyr, $V_0 = 1.5$, $V_1 = 0.2$, $\tau_d = 20$ kyr, we get $V = 0.4$ and $\Psi = 2$. As we
 418 have established before, for VCV18-1 model, for $\tau = 50$ kyr, $V = 0.63$, $\Phi = 2$. Therefore, we can talk
 419 about *physical similarity* between VCV18-1 and LP22 models in terms of reasonably close V -numbers.

420 The similarity between VCV18-1 and LP22 models becomes very visual in Fig. 2(f) describing the
 421 positive-versus-negative feedback loops. It can be observed that in VCV18-1 model, during much of ice
 422 growth, positive and negative feedbacks also completely compensate each other. In fact, we can consider
 423 LP22 model as being an approximation of the VCV18-1 feedback loop by a triangle. Indeed, the LP22
 424 positive and negative feedbacks compensate each other during ice advance and when the critical value of
 425 v (i.e., V_0) is achieved, the system instantly migrates to the single dominant negative feedback. Simply
 426 speaking, these two models are as similar as the shape of the LP22 feedback-loop triangle in Fig. 2(f) is
 427 similar to the shape of the VCV18-1 feedback loop, since the V -number is the ratio of its horizontal
 428 dimension to its vertical dimension. **This similarity can be illustrated even further if we compare explicit**
 429 **scaling laws (29) and (47). For this purpose, we solved equations (22) – (23) changing parameter c and**
 430 **thus gradually changing the balance between positive and negative feedbacks. The results of these**
 431 **additional experiments are presented in Fig. 3 together with the P -scaling law (47) that can be estimated**
 432 **in a very straightforward manner: $P \sim V_0 \tau_g + \tau_d$, or $\frac{P}{\tau_g} \sim V_0 + \tau_d/\tau_g$, that is finally $\frac{P}{\tau_g} \sim V_0 + V$. It can be**
 433 **observed that the LP22 scaling law resembles well the VCV18-1 scaling-law trendline thus supporting**
 434 **our assertion that we can consider LP22 model as an approximation of the VCV18-1 model.**

435

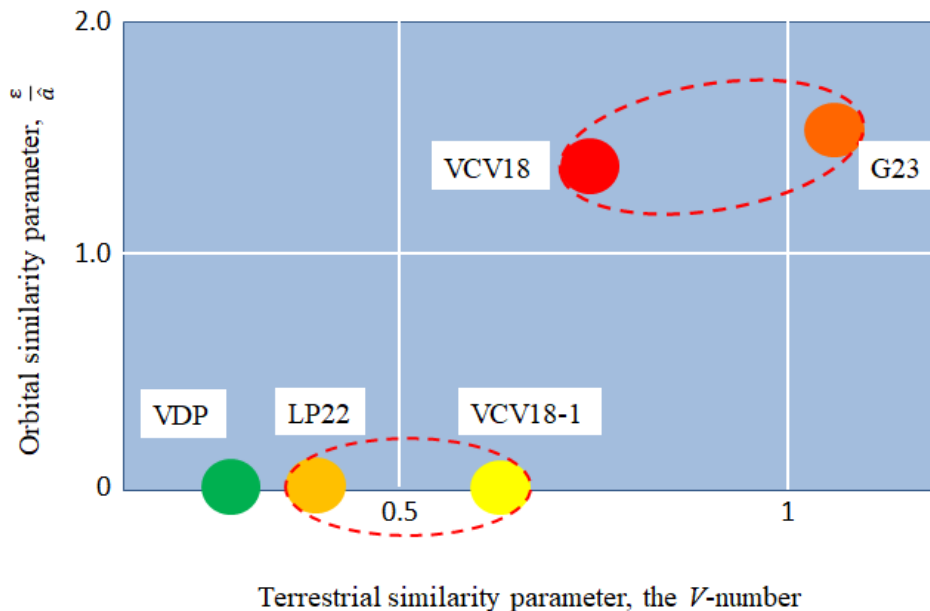


436
 437 **Fig. 3.** The P -scaling laws of VCV18-1 (blue, the dotted line is its trendline) and of LP22 (brown)
 438 models.

439 **3. Conclusions**

440
 441 Nikolai Gogol would have said: “A magic apple tree may grow golden apples...but not pears”.
 442 Phenomenological models may not be bounded by a specific physics but they have to be consistent with
 443 the laws of physics. The concept of physical similarity is what allows us to be vigilant about such
 444 consistency. Accordingly, we started our presentation with the question: Are phenomenological
 445 dynamical paleoclimate models physically similar to Nature? We demonstrated that, though they may be
 446 remarkably accurate in reproducing empirical time series, this is not sufficient to claim physical similarity
 447 with Nature until similarity parameters are considered. We illustrated such a process of diagnostics of
 448 physical similarity by comparing three phenomenological dynamical paleoclimate models with the more
 449 explicit model that played the role of parent dynamical system. Though the nomination of the VCV18
 450 model to serve as a proxy of the parent dynamical system can, indeed, be questioned, and the
 451 developments of better proxies should be encouraged, we nevertheless believe that the diagnostics of
 452 physical similarity, we have described, should become a standard procedure before a phenomenological
 453 model can be utilized for interpretations of historical records or for future predictions. In other words,
 454 claiming a model to be a phenomenological one is not an indulgence but a liability.

455 The results of the analysis are summarized in Fig. 4. Here, the physical similarity is visualized as
 456 proximity between the models in the $(\frac{\varepsilon}{\bar{a}}, V)$ space.



457
 458 **Figure 4.** Physical similarity diagnostics for the obliquity-period doubling of the VCV18 and G23 models
 459 and for 100-kyr auto-oscillations of the VDP, LP22, and VCV18-1 models. The physical similarity is
 460 visualized here as proximity between the models in the $\frac{\varepsilon}{\bar{a}} - V$ space. Red dashed ovals embrace models
 461 which we found to be physically similar and therefore they are in a reasonably close proximity in the
 462 diagram.

463
 464 LP22 and G23 models can be considered to be physically similar to particular versions of the VCV18
 465 model, which amounts to saying that they may be derived from basic laws of physics under some
 466 reasonable physical assumptions. These findings boost the physical viability of these phenomenological
 467 models, but this is not unconditional and there are clear boundaries of these phenomenological models
 468 physical legitimacy. For LP22, the ratio of feedback’s amplitudes, the V-number, is also the ratio of

469 timescales. For the Late Pleistocene and for the Early Pleistocene, all timescales are likely to be different
470 and physical similarity therefore would need to be re-examined for each period separately. For G23
471 model, physical similarity was found only for the obliquity-range forcing.

472 Generally speaking, our observation that the VDP model is not physically similar to the simplified
473 version of the VCV18 model is not a final verdict. It is indeed an indication that VDP is not based on *ice*
474 physics, but there may be other physical phenomena that may provide physical legitimacy to VDP model.
475 We are a bit skeptical though that such phenomenon can easily be found, because it would need to be
476 constrained in the same way as VDP is. Specifically, its ratio of positive and negative feedbacks must to
477 be fixed to a specific value that never changes.

478 Fig. 4 clearly shows that VDP, LP22 and even VCV18-1 models are not physically similar to the
479 VCV18 model. Indeed, VDP, LP22, and VCV18-1 scaling laws do not have a similarity parameter that is
480 a ratio of orbital and terrestrial forcing amplitudes and therefore these models are located on the
481 horizontal axis of Fig. 4 diagram. Interestingly, VDP, LP22, and VCV18-1 models are located to the left
482 from the VCV18 and G23 models thus revealing more prominent role of their negative feedbacks. As it
483 is clearly displayed in the positive-versus-negative feedback diagrams of Fig. 2, it happens because the
484 mechanism of ice disintegration in these two groups of models is based on very different physics. In
485 VCV18-1, VDP, and LP22 models, the disintegration of ice sheets happens when the negative feedback
486 suddenly becomes dominant and destroys an ice sheet. In both VCV18 and G23 models, the
487 disintegration is due to the additional, fast, positive feedback, which is small during most of the ice-
488 growth period, but eventually becomes strong enough to boost the orbital forcing that attempts ice
489 destruction.

490 Importantly, we arrived to the above conclusions without making any effort to somehow artificially
491 minimize (or maximize) a “distance” between the models in Fig. 4 (though it may be an interesting
492 separate exercise) but used instead, for the most part, published reference values of model parameters.

493 Also, it should not be forgotten that Fig. 4 is just our attempt to present most important results
494 visually in one diagram. Since it is 2-dimensional, it demonstrates similarity (or its absence) in terms of
495 two similarity parameters, the ratio of astronomical forcing amplitude to the terrestrial growth rate and the
496 positive-to-negative feedback ratio. These two dimensions wouldn’t be sufficient if more similarity
497 parameters need to be visually presented. For example, equation (21) shows that G23 model depends on
498 the similarity parameter that is the ratio of the orbital period T to the internal timescale $1/a$. This similarity
499 parameter is absent in the VCV18 model. This absence of similarity, indeed, is not apparent from Fig. 4,
500 but it is discussed in paragraph 2.2.

501 As a final conclusion, we agree with Saltzman’s (2002) proposal that “the essential slow physics is to
502 be sought in the low-order models.” We observe though that “essential slow physics” that can be derived
503 from phenomenological models is limited to orbital and terrestrial timescales, to ratios of amplitudes of
504 orbital and terrestrial forcings, and to ratios of amplitudes of positive and negative feedbacks. This is as
505 much as phenomenological models can offer, and therefore, we deviate from Saltzman’s (2002) further
506 idea that more explicit models should be tuned to satisfy a best phenomenological model. Instead, we
507 propose to use available physical models for diagnostics of physical-similarity hypothesis that needs to be
508 either confirmed or rejected. **Specifically, physical models have to be explored to identify additional
509 dimensionless quantities playing key roles in their dynamics. Further, finding values for these quantities
510 should be a central objective of future research into glacial-interglacial dynamics.**

511 Of course, encoding empirical data in a simple mathematical statement will always remain a tempting
512 possibility. As Grigory Barenblatt (2003) said “Applied mathematics is the *art* of constructing
513 mathematical models of phenomena in nature...” This means that there are no strict rules on how a piece
514 of mathematical “art” needs to be produced. Therefore, we do not attempt here to discourage our fellow
515 “artists” from alluding to phenomenological models. Our goal instead was to remind them about
516 phenomenological models’ limitations and to suggest how these limitations may be addressed.

517

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519 wrote the first draft of the manuscript. The authors jointly discussed the findings and contributed equally
520 to the editing of the manuscript.

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