Response to Anonymous Referee #1

Dear Anonymous Referee #1,

We are grateful for your insightful review that will help us to improve the manuscript. Your *verbatim* comments are below (in bold), each followed by our response.

General comments: This manuscript has the potential to provide an important contribution to Ice Age modelling.

Response: We are encouraged by your recognition of such important potential.

General comments (cont'd): Building on their previous work (Verbitsky & Crucifix, 2020; 2022), the authors investigate the physical similarity of different models using dimensional analysis. A key finding is that although all these models are structurally dissimilar, they share a dependence of the modeled periodicity on the dimensionless "V" parameter or the ratio between positive and negative feedbacks in the system.

Response: Your observation is correct: Indeed, finding the key similarity parameter that allowed us to quantitatively compare phenomenological models with the model that we nominated to serve as the proxy of the parent dynamical system is very important. Another key finding, that you have not mentioned, is that not all phenomenological models have actually "passed" the similarity test and therefore did not demonstrate a link with known physical assumptions. This finding has in our view philosophical-level consequences: successful replication of empirical time series is not sufficient to claim physical similarity with Nature.

Having said all this, I do think that the authors will need to address the issues discussed below before the manuscript will be ready for publication.

Specific comments: I have two major concerns that the authors will need to address in a compelling manner:

1) The authors make it seem as if they are comparing models to Nature when they are really comparing different models to each other. And yes, even the Verbitsky et al. (2018) model is not the same thing as Nature. The authors should be transparent about this, which starts with the title. I suggest changing it from "Do phenomenological dynamical paleoclimate models have physical similarity with Nature? Seeming, but not all of them" to something like "Structural similarities and differences between paleoclimate models of glacial-interglacial dynamics". Then, the Abstract and Introduction could be framed around questions such as "To which extent are different paleoclimate models of glacial-interglacial dynamics physically similar?" and "Are there any shared dimensionless quantities playing key roles in all these models? If so, then finding values for these quantities should be a central objective of future research into glacial-interglacial dynamics."

Response: Your concern is fair and it is well understood. In fact, we tried hard to be very transparent in that regard: (a) We introduced the notion of the parent dynamical system (lines 63-69), (b) then we formulated requirements to it and introduced the VCV18 model as a candidate, a proxy, for the parent system (lines 108-113, 132-143), (c) we even suggested that "the nomination of the VCV18 model to serve as a proxy of the parent dynamical system can, indeed, be questioned, and the developments of better proxies should be encouraged" (lines 427-429). We agree with you though that there is still room for improvement, therefore…

Action: Your questions "To which extent are different paleoclimate models of glacial-interglacial dynamics physically similar?" and "Are there any shared dimensionless quantities playing key roles in all these models? If so, then finding values for these quantities should be a central objective of future research into glacial-interglacial dynamics" are well formulated and we will definitely discuss them in the introduction and conclusions sections of the revised paper. We are a bit reluctant though to accept your new title suggestion because, in our opinion, it shifts the focus of the paper. We believe that it is not so much about how similar models are, but about how similar are models, that are merely a statistical description of the data (phenomenological models), and a model that was derived from the basic laws of physics. We will make sure though that the Nature in the title will become quote-unquote "Nature".

2) The authors use the Buckingham pi theorem to answer the identify the dimensionless parameters affecting the period of the system in each of the models. All well and good, but what do we really learn from this about glacial-interglacial dynamics? I think much more insight could be gained if the authors would identify the actual relationships between the parameters and the period of the model systems (i.e., the psi and phi functions). It should not be too difficult to find decent approximations of these relationships through simulations, given that the models under consideration are rather simple and computationally cheap to run. Such an exercise would also give much deeper insight about the physical similarity between the different models. For example, the period may depend on "V" in two models, but the scaling may be V^2 in one model and V^3 in the other. Then, one can ask where these different scalings originate from and whether any of these scalings could be tested against observational data.

Response: Your observation is correct – we do not provide the most explicit form of the scaling laws, specifically, we do not convert

$$\Pi = \Phi(\Pi_1, \Pi_2, \dots, \Pi_i, \dots, \Pi_m) \qquad \text{(AC1)}$$

into

$$\Pi = {\Pi_1}^{\alpha_1} {\Pi_1}^{\alpha_2} \dots {\Pi_1}^{\alpha_m} \qquad \text{(AC2)}$$

Also, we do not convert

$$\pi = \Psi(\pi_1, \pi_2, \dots, \pi_i, \dots, \pi_q) \qquad \text{(AC3)}$$

into

$$\pi = {\pi_1}^{\beta_1} {\pi_1}^{\beta_2} \dots {\pi_1}^{\beta_q} \qquad \text{(AC4)}$$

We limited our scope to the demonstration that in some models

$$\Psi \approx \Phi \qquad \text{(AC5)}$$

when

$$\pi_i \approx \Pi_i, \qquad \text{(AC6)}$$

i.e., π_i-physics in the model is as significant as the Π_i-physics of "Nature". It allowed us to "delineate a model that is merely a statistical description of the data, from a model that can be claimed to have a link with known physical assumptions" (lines 20-22). To our knowledge, *this result is novel.*

Importantly, *the condition (AC6) can also be considered as a constraint on the parameterization structure of a model, and as such, it is indeed also a new insight.*

Though we agree with you that the finding of explicit scaling laws (AC2) and (AC4) is an important exercise, we believe that this should be a separate study because it may be a bit more challenging than it seems to be. For example, gradual increase of the *V*-number in the VCV18 model includes a bifurcation from the obliquity period to the obliquity-period doubling (Verbitsky and Crucifix, 2020). Moreover, the *V*-number in the VCV18 model is a conglomerate similarity parameter, composed by 5 similarity parameters and, though the bifurcation is imminent, the critical value of the *V*-number that starts a bifurcation depends on which specific similarity parameters produce the *V*-number change. For example, the bifurcation that is caused by increased *V*-number due to the intensification of the positive feedback and caused by changes of the governing parameter responsible for the positive feedback may start at the critical *V*-number value that is different from the critical *V*-number value caused by the weakening of the negative feedback (Verbitsky, 2022). Therefore, to get a complete picture, the number of experiments needed for spanning the full parameter space can be overwhelming.

However, to address your concern, a few steps in this direction can be done, indeed, without much of computer power. First, we have run a few additional experiments with the VCV18-1 model:

$$\frac{dH}{dt} = \hat{a} + \kappa H^4 - c\theta \qquad (22)$$

$$\frac{d\theta}{dt} = \frac{H^4 - \theta}{H/\hat{a}} \qquad (23)$$

Specifically, we numerically measured the dimensionless period of auto-oscillations (*P*-scaling law) $\frac{P}{\tau} = \Phi(V)$, changing parameter *c*, and thus gradually changing the balance between positive and negative feedbacks. The results of these additional experiments are presented in Figure AC1 together with the *P*-scaling law for LP22 model $\frac{P}{\tau_g} = \Psi(V_0, V)$ that can be estimated in a very straightforward manner:

$P = V_0\tau_g + \tau_d$, or

$\frac{P}{\tau_g} = V_0 + \tau_d/\tau_g$, or

$$\frac{P}{\tau_g} = V_0 + V \qquad (AC7)$$

And finally, as an illustration, we also made a few experiments with the VCV18 model, gradually changing the *V*-number and measuring the corresponding *P*-scaling law. For this specific illustration, the *V*-number was modified by changing the strength of the positive feedback through the coefficient γ in the equation (10). The results are also presented in Figure AC1.

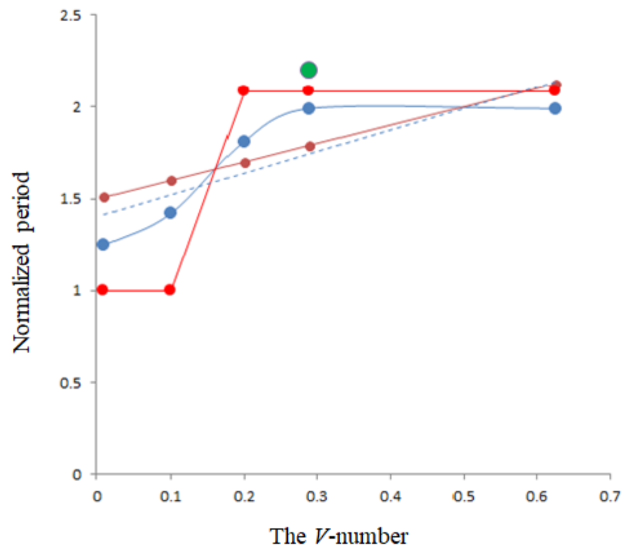

Fig. AC1. The *P*-scaling laws of VCV18-1 (blue, the dotted line is its trendline) and of LP22 (brown) models. The green dot marks the scaling law for the VDP model. The red line illustrates a bifurcation from the obliquity period to the obliquity-period doubling in the VCV18 model.

Interestingly, though the VCV18-1 scaling law $\frac{P}{\tau} = \Phi(V)$ and LP22 scaling law $\frac{P}{\tau_g} = \Psi(V_0, V)$ were produced absolutely independently, the LP22 scaling law coincides almost perfectly with the VCV18-1 scaling-law trendline. This closeness of the VCV18-1 and LP22 scaling laws supports our previous assertion that we can consider LP22 model as an approximation of the VCV18-1 (lines 407 – 415). The new figure also shows that better articulated positive feedbacks (i.e., increased *V*-number) lead to longer periods of both auto-oscillations and periods of the system response to the orbital forcings.

Action: We will include parts of the above conversation in the paper. In doing so, we will rely on our editor advice to reasonably constrain the divergence from the original scope.

Technical corrections

l. 16: "...similar with the..." -> "...similar to the..."

l. 55: "Dynamical system's theory tell us why..." -> "Dynamical Systems Theory tells us why..."

l. 59: "...phemenon..." -> "...phenomenon..."

l 79: "...parameters, let say Pi_1, is..." -> "....parameters, say Pi_1, is..."

l. 375: "...according to pi-theorem:" -> "...according to the pi-theorem:"

Action: Thank you! All corrections will be taken care of.

Respectfully,

Mikhail Verbitsky and Michel Crucifix

References

Verbitsky, M. Y.: Inarticulate past: similarity properties of the ice–climate system and their implications for paleo-record attribution, Earth Syst. Dynam., 13, 879–884, https://doi.org/10.5194/esd-13-879-2022, 2022.

Verbitsky, M. Y. and Crucifix, M.: π-theorem generalization of the ice-age theory, Earth Syst. Dynam., 11, 281–289, https://doi.org/10.5194/esd-11-281-2020, 2020.