1 Derivation of wavelet transformation normalisation factor

Consider a finite time series, expressed continuously as a function of time:

$$x = x(t) \quad | \quad t \in [0, T] \tag{1}$$

We define a Morlet wavelet as

$$\phi(a, t) \;=\; b \exp\left(-i\omega_0 \frac{t}{a}\right) \exp\left(-\frac{t^2}{2a^2}\right) . \tag{2}$$

The choice of the normalisation parameter b depends on the application. Typically we want to normalise the individual wavelet and choose $b = \pi^{-0.25} a^{-0.5}$. In this case we get

$$\phi(a, t) \;=\; \pi^{-0.25} a^{-0.5} \exp\left(-i\omega_0 \frac{t}{a}\right) \exp\left(-\frac{t^2}{2a^2}\right) , \tag{3}$$

$$\int_{t=-\infty}^{\infty} \phi(a, t)\phi^*(a, t) \;=\; \pi^{-0.5} a^{-1} \int_{t=-\infty}^{\infty} \exp\left(-\frac{t^2}{a^2}\right) . \tag{4}$$

If we define $\sigma = a/\sqrt{2}$ we can use the normalisation of the Gaussian curve:

$$\int_{t=-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\sigma^2}\right) \;=\; 1 \tag{5}$$

$$\int_{t=-\infty}^{\infty} \exp\left(-\frac{t^2}{2\sigma^2}\right) \;=\; \sigma\sqrt{2\pi} \tag{6}$$

and obtain

$$\int_{t=-\infty}^{\infty} \phi(a, t) \;=\; \pi^{-0.5} a^{-1} \sigma\sqrt{2\pi} = 1 . \tag{7}$$

Let's Fourier transform the wavelet. To do so, we Fourier-transform parts of the formula first:

$$\exp\left(-\frac{t^2}{2a^2}\right) \;=\; \int_{\omega=-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} a\sqrt{2\pi} e^{\frac{-\omega^2 a^2}{2}} , \tag{8}$$

$$\exp\left(-i\frac{\omega_0}{a} t\right) \;=\; \int_{\omega=-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} 2\pi\delta(\omega - \frac{\omega_0}{a}) , \tag{9}$$

$$\tag{10}$$

Since $\phi(a,t)$ is a product, we can write $\phi(a,\omega)$ as a convolution:

$$\phi(a,t) = \pi^{-0.25}a^{-0.5} \int\limits_{\omega=-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{1}{2\pi} \left(a\sqrt{2\pi} e^{\frac{-\omega^2 a^2}{2}} \right) * \left(2\pi\delta(\omega - \frac{\omega_0}{a}) \right) \quad (11)$$

$$= \pi^{0.25}\sqrt{2a} \int\limits_{\omega=-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \left(e^{\frac{-(a\omega)^2}{2}} \right) * \left(\delta(\omega - \frac{\omega_0}{a}) \right) \quad (12)$$

$$= \pi^{0.25}\sqrt{2a} \int\limits_{\omega=-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} e^{\frac{-(a\omega-\omega_0)^2}{2}} \,. \quad (13)$$

So we obtain the Fourier transform as

$$\tilde{\phi}(a,\omega) = \pi^{0.25}\sqrt{2a} e^{\frac{-(a\omega-\omega_0)^2}{2}} \,. \quad (14)$$

In practice, we want to calculate a convolution of the wavelet and the time series:

$$W(a,t) = \int\limits_{\tau=-\infty}^{\infty} d\tau x(\tau)\phi(a,t-\tau) \,. \quad (15)$$

Since this is a convolution, we can write it as a product in Fourier space:

$$W(a,t) = \int\limits_{\omega=-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{x}(\omega)\tilde{\phi}(a,\omega) \,. \quad (16)$$

In practice, we do the wavelet transform in the following way: We start from a time series $x_1, \ldots, x_N$ which is the function $x(t)$ sampled with an equidistant time step Δt. It is padded with zeros (between $N/\sqrt{2}$ and $N\sqrt{2}$ such that the resulting length is a power of 2). Then $\tilde{x}_k$ is the fast Fourier transform of this padded time series.

Next we define a set of temporal scales a. For each of these a, we calculate the product in Fourier space as

$$\tilde{W}(a,\omega) = \tilde{x}(\omega)\pi^{0.25}\sqrt{2a} e^{\frac{-(a\omega-\omega_0)^2}{2}} \,. \quad (17)$$

Back-transformation then gives the wavelet amplitude $W(a,t)$.

Let's take a look at the units which we get. Assume for simplicity that x has the unit m. Time is measured in seconds, so a has the unit s and ω has the unit s^{-1}. The relative scale ω_0 is dimensionless.

Equation (3) states that $\phi(a,t)$ has the unit s$^{-0.5}$. So from equation (15) we can derive a unit for $W(a,t)$ which is m s$^{0.5}$ (note that dt has a unit of s). This is very impractical since this unit has no direct physical meaning.

Let's consider alternative normalisations. For example, let $x(t)$ be a harmonic oscillation with a period matching that of the Wavelet and an amplitude of 1 m:

$$x(t) = (1\,\mathrm{m}) \exp\left(-i\frac{\omega_0}{a} t \right) \,. \quad (18)$$

We may want the wavelet to have an amplitude of $x_0 = 1$ m as well. In this case, we obtain:

$$\begin{aligned} x_0 = |W(a,t)| &= \left| \int\limits_{\tau=-\infty}^{\infty} d\tau \, x(\tau) \, \phi(a, t-\tau) \right| & (19) \\ &= \left| \int\limits_{\tau=-\infty}^{\infty} d\tau \, x_0 \exp\left(-i\frac{\omega_0}{a}\tau\right) \, b \exp\left(-i\omega_0 \frac{t-\tau}{a}\right) \exp\left(-\frac{(t-\tau)^2}{2a^2}\right) \right| \\ &= \left| x_0 b \exp\left(-i\omega_0 \frac{t}{a}\right) \right| \left| \int\limits_{\tau=-\infty}^{\infty} d\tau \, \exp\left(-\frac{(t-\tau)^2}{2a^2}\right) \right| \\ &= |x_0 b| \left| a\sqrt{2\pi} \right| \\ b &= a^{-1}(2\pi)^{-0.5} & (20) \end{aligned}$$

However, in reality, we only have the real part of the signal. Since both the real and the imaginary part will contribute the same to the amplitude, we need to multiply by a factor of two. Yes, it is a factor of two and not of $\sqrt{2}$ because we never square the signal $x(t)$ so $W(a,t)$ depends linearly on $x(t)$. We end up with

$$b = a^{-1}\pi^{-0.5}\sqrt{2} \,. \tag{21}$$

This means the wavelet is defined as

$$\phi(a,t) = \frac{\sqrt{2}}{\sqrt{\pi}a} \exp\left(-i\omega_0 \frac{t}{a}\right) \exp\left(-\frac{t^2}{2a^2}\right) \,. \tag{22}$$

If we compare to the previous value of

$$b_{old} = \pi^{-0.25} a^{-0.5} \tag{23}$$

we find that we need to correct by a factor of $\pi^{-0.25}a^{-0.5}\sqrt{2}$ and obtain

$$\tilde{W}(a,\omega) = \tilde{x}(\omega) 2 e^{\frac{-(a\omega-\omega_0)^2}{2}} \,. \tag{24}$$